

A REMARK ON THE ABEL-JACOBI MORPHISM FOR THE CUBIC THREEFOLD*

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ABSTRACT. Let X be a smooth cubic threefold and $J(X)$ be its intermediate Jacobian. We show that there exists a codimension 2 cycle Z on $J(X) \times X$ with Z_t homologically trivial for each $t \in J(X)$, such that the morphism $\phi_Z : J(X) \rightarrow J(X)$ induced by the Abel-Jacobi map is the identity. This answers positively a question of Voisin in the case of the cubic threefold.

1. INTRODUCTION

A classical theorem of Abel states that

Theorem 1.1. *Let C be a smooth projective complex curve. Then each fiber of the Abel-Jacobi map*

$$AJ_C : \text{Sym}^d C \rightarrow J(C)$$

is a projective space for all $d \geq g(C)$. Moreover, the induced morphism

$$CH_0(C)_{\text{num}} \rightarrow J(C)$$

is an isomorphism.

In particular, the geometry of the fibers of the Abel-Jacobi map for curves is well understood.

For higher dimensional varieties, the work of Bloch-Ogus [2], Bloch-Srinivas [3], Merkurjev-Suslin [11] and Murre [12] leads to the following theorem, which can be regarded as the higher dimensional generalization of the second assertion of Theorem 1.1.

Theorem 1.2. ([12]) *Let X be a smooth projective complex variety such that $CH_0(X)$ is supported on a curve. Then $CH^2(X)_{\text{hom}} = CH^2(X)_{\text{alg}}$ and the Abel-Jacobi map induces an isomorphism*

$$AJ_X : CH^2(X)_{\text{hom}} \rightarrow J(X) := H^3(X, \mathbb{C}) / (F^2 H^3(X) \oplus H^3(X, \mathbb{Z})).$$

In the present note, we will consider the case where X is a rationally connected threefold, so that $CH_0(X) = \mathbb{Z}$ is supported on a point and $CH^2(X) = CH_1(X)$.

Since the group $CH_1(X)_{\text{alg}}$ does not have the structure of an algebraic variety, one has to be careful when stating that AJ_X is algebraic. In fact, $CH_1(X)_{\text{alg}}$ is an inductive limit of quotients of algebraic varieties by an equivalence relation, and to say that the morphism AJ_X is algebraic means by definition that for any smooth projective variety Y and any codimension 2 cycle Z on $Y \times X$ with $Z_y \in CH^2(X)_{\text{hom}}$ for any $y \in Y$, the induced morphism

$$\phi_Z : Y \rightarrow J(X), \quad \phi_Z(y) = AJ_X(Z_y),$$

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is a morphism of algebraic varieties, which will be called *the Abel-Jacobi morphism*.

An important observation made by Voisin is that, despite the similarity between Theorem 1.2 and Abel's theorem 1.1, there are substantial differences between 1-cycles on threefolds with small CH_0 and 0-cycles on curves, which she relates to the geometry of the fibers of the Abel-Jacobi morphisms.

In fact, the following two questions are proposed in [13].

Question 1.3. *Let X be a smooth projective threefold such that $AJ_X : CH_1(X)_{alg} \rightarrow J(X)$ is surjective. Is there a codimension 2 cycle Z on $J(X) \times X$ with $Z_t \in CH^2(X)_{hom}$ for any $t \in J(X)$ such that the Abel-Jacobi morphism*

$$\phi_Z : J(X) \rightarrow J(X), \quad \phi_Z(t) = AJ_X(Z_t)$$

is the identity?

As remarked by Voisin, Question 1.3 has a positive answer if the Hodge conjecture holds true for degree 4 *integral* Hodge classes on $J(X) \times X$.

Question 1.4. *For which threefolds X is the following property satisfied?*

There exist a smooth projective variety Y and a codimension 2 cycle Z on $Y \times X$ with $Z_y \in CH^2(X)_{hom}$ for any $y \in Y$, such that the Abel-Jacobi morphism $\phi_Z : Y \rightarrow J(X)$ is surjective with rationally connected general fiber.

It is known that Question 1.4 has a positive answer for smooth cubic threefolds [9], [10] and smooth complete intersections of two quadrics in \mathbb{P}^5 [4]. It was proved in [13] that if Question 1.4 has a positive answer for X and the intermediate Jacobian $J(X)$ admits a 1-cycle Γ such that $\Gamma^{*g} = g!J(X)$ in $CH_g(J(X)) = \mathbb{Z}$, where $g = \dim J(X)$, then Question 1.3 also has a positive answer for X . In particular, if the intermediate Jacobian of X is isomorphic to the Jacobian of a curve, then Question 1.3 has a positive answer for X if Question 1.4 does. Therefore, Question 1.3 has a positive answer for smooth complete intersections of two quadrics in \mathbb{P}^5 .

Unfortunately, there are very few rationally connected threefolds whose intermediate Jacobians are not Jacobians and for which the existence of a cycle Γ as above is known (this is a very classical question in the case of general cubic threefolds, and is equivalent in this case to the algebraicity of the so-called minimal class $\frac{\Theta^4}{4!}$ which is an integral Hodge class on $J(X)$). Question 1.3 needs therefore other approaches.

In this note we give a positive answer to Question 1.3 for any smooth cubic threefold. For the properties of the intermediate Jacobian of the cubic threefold, see [5]. The key point of our proof lies in the observation that the moduli space of stable sheaves of rank 2 with Chern numbers $c_1 = 0, c_2 = 2, c_3 = 0$ on a smooth cubic threefold is fine.

We will work over the complex number field \mathbb{C} .

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2. THE MAIN RESULT

In this section, we state and prove the main result of this note.

Theorem 2.1. *Let X be a smooth cubic threefold. Then there exists a codimension 2 cycle Z on $J(X) \times X$ with $Z_t \in CH^2(X)_{\text{hom}}$ for any $t \in J(X)$, such that the induced Abel-Jacobi morphism $\phi_Z : J(X) \rightarrow J(X)$ is the identity.*

We will need a sufficient condition for an open subset of the moduli space of stable sheaves on a smooth projective variety to be fine.

Let X be a smooth projective variety. Recall that the Grothendieck group modulo numerical equivalence $K_{\text{num}}(X)$ is defined to be $K(X)/\equiv$, where two classes x and y in $K(X)$ are said to be numerically equivalent (notation $x \equiv y$), if the difference $x - y$ is contained in the radical of the quadratic form

$$(a, b) \mapsto \chi(a \cdot b) = \int_X \text{ch}(a)\text{ch}(b)\text{td}(X)$$

(cf. [7]). Now fix a class $c \in K_{\text{num}}(X)$. Let P be the associated Hilbert polynomial, M^s be the moduli space of stable sheaves on X and $M(c)^s \subset M^s$ be the open and closed part parametrizing stable sheaves of numerical class c .

Theorem 2.2. ([8, Th.4.6.5]) *If the greatest common divisor of all numbers $\chi(c \cdot \mathcal{F})$, where \mathcal{F} runs through some collection of coherent sheaves on X , is equal to 1, then there is a universal sheaf on $M(c)^s \times X$.*

Theorem 2.3. *Let X be a smooth cubic threefold. Then the moduli space $M_X^s(2; 0, 2)$ of stable sheaves of rank 2 with Chern numbers $c_1 = 0, c_2 = 2, c_3 = 0$ on X is fine. Equivalently, there exists a universal sheaf on $M_X^s(2; 0, 2) \times X$.*

Proof. Let c be the numerical class of a locally free, stable sheaf \mathcal{E} of rank 2 on X . Recall that $\text{Pic}(X) = \mathbb{Z} \cdot h$, where h is the class of a hyperplane section of X . The Chow group of 1-cycles on X modulo algebraic equivalence $A_1(X) = \mathbb{Z} \cdot l$, where l is the class of a line on X . Note that $h^2 \equiv 3l$. Since X is Fano, $\text{CH}_0(X) = \mathbb{Z} \cdot pt$. Then $\text{ch}(c) \equiv \text{ch}(\mathcal{E}) = 2 - 2l$. Since $c_1(\mathcal{T}_X) = 2h$ and $c_2(\mathcal{T}_X) = 12l$, $\text{td}(X) = 1 + h + 2l + pt$. It is easy to see that $\text{ch}(\mathcal{O}_X(1)) = 1 + h + \frac{3}{2}l$. Then we compute that $\chi(c \cdot \mathcal{E}) = -4$,

$$\begin{aligned} \chi(c \cdot \mathcal{O}_X(1)) &= \int_X \text{ch}(c) \cdot \text{ch}(\mathcal{O}_X(1)) \cdot \text{td}(X) \\ &= \int_X (2 - 2l) \cdot (1 + h + \frac{3}{2}l) \cdot (1 + h + 2l + pt) = 5. \end{aligned}$$

Obviously, the greatest common divisor of $\chi(c \cdot \mathcal{E})$ and $\chi(c \cdot \mathcal{O}_X(1))$ is equal to 1. Then Theorem 2.2 implies that there exists a universal sheaf on $M_X^s(2; 0, 2) \times X$. \square

Proof of Theorem 2.1. It is shown in [6], [9] (see also [1]) that the morphism $\phi : \text{Hilb}^{5t}(X) \rightarrow J(X)$ factorizes through the birational morphism $c_2 : M_X(2; 0, 2) \rightarrow J(X)$. Moreover, letting M_0 be the open subset of $M_X(2; 0, 2)$ parametrizing locally free stable sheaves, the restricted morphism $M_0 \rightarrow J(X)$ is an open immersion. Now we regard M_0 as an open subset of $J(X)$. Let Z' be the closure in $J(X) \times X$ of a global section of $\mathcal{E}|_{M'_0 \times X}$, where M'_0 is an open subset of M_0 over which such a transverse section exists, and $Z = Z' - J(X) \times C_0$, where C_0 is a quintic elliptic curve on X . Then the induced Abel-Jacobi morphism $\phi_Z : J(X) \dashrightarrow J(X)$ is the identity, since by construction it induces the natural inclusion on M'_0 .

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